Noncommutative geometry in mixed characteristic

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The basic setup

- $\mathbb{F}_p$ - the finite field with p (prime) elements;
- $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ - the p-adic integers, $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$ the p-adic numbers;
- Lift $\mathbb{F}_p$-algebras to "topological" $\mathbb{Z}_p$-algebras;
- Construct invariants of topological $\mathbb{Z}_p$-algebras that only depend on their reduction mod p.

Background from arithmetic geometry

- Let A be the coordinate ring of a smooth affine variety over $\mathbb{F}_p$;
- The number of points of the variety (in finite extensions of $\mathbb{F}_p$) can be computed using an appropriate variant of de Rham cohomology in positive characteristic via the Frobenius map $x \mapsto x^p$;
- This is used to obtain structural information about the zeta function of the variety (Weil conjectures);
- Algebraic de Rham cohomology is badly behaved in positive characteristic.

How to do de Rham theory in positive characteristic?

- Need to decide whether or not you want homotopy invariance;
- For computations in the noncommutative setting, we want homotopy invariance.

Theorem (Grothendieck)

Any smooth, finite-type commutative $\mathbb{F}_p$-algebra A has a smooth, finite-type $\mathbb{Z}_p$-algebra lifting R.

- Complete the $\mathbb{Z}_p$-algebra lifting R "suitably";
- Banach completion does not work - instead, take dagger completion $R^\dagger$.
- Define $H^*(A) := H^*(R^\dagger \otimes \mathbb{Q}_p)$. This does not depend on choices of liftings (Monsky-Washnitzer-Berthelot).

Noncommutative de Rham theory

- Periodic cyclic homology generalises de Rham cohomology to the setting of noncommutative algebras;
- Two issues emerge - no analogue of Grothendieck's result, and we need a coordinate free description of dagger completion (for noncommutative algebras);
- Need to choose a lifting and work with periodic cyclic homology relative to such a choice;
- Analytic cyclic homology resolves this issue, but is hard to compute in general.

Smooth subalgebras of C^*-algebras

- To get a well-behaved cohomology theory for C^*-algebras, we work with 'smooth' subalgebras of C^*-algebras (eg: $C^\infty(M) \subseteq C(M)$);
- In the noncommutative setting, these are dense embeddings $A \subseteq B$, where B is an inductive limit of Banach algebras and the inclusion preserves bornological spectral radii of bounded subsets;
- To construct these, consider a dense subalgebra $A \subseteq B$ of a (local) Banach algebra. Then $A^{\infty,B} := \bigcup_{\varrho(S)<1} \mathbb{R}\langle S^\infty \rangle$, where

$$\langle S^\infty \rangle := \{\sum_{i=1}^{\infty} \lambda_i x_{i,1} \dots x_{i,l(i)} : \sum_{i=1}^{\infty} |\lambda_i| < \infty\}.$$

- Examples - G a finitely genrated discrete group, $\mathcal{S}(G) \subseteq l^1(G)$, $\mathcal{S}(\mathbb{T}^2_\theta) \subseteq C^*(\mathbb{T}^2_\theta)$.

Smooth subalgebras of nonarchimedean Banach algebras

- The same strategy can be used to define 'smooth subalgebras' of p-adically complete Banach algebras;
- Take a $\mathbb{Z}_p$-subalgebra R of a Banach $\mathbb{Z}_p$-algebra B and enlarge its bornology to include submodules of the form $\sum_{n=0}^{\infty} p^n S^{n+1}$;
- The completion in this bornology is called the dagger completion $R \to R^\dagger$ of R.

Periodic cyclic homology

- Cortiñas-Cuntz-Meyer-Tamme (2017): R a smooth, finite-type $\mathbb{Z}_p$-algebra, $\mathrm{HP}_*(R^\dagger \otimes \mathbb{Q}_p) = \bigoplus_{n\in\mathbb{Z}} H^{*+2n}(A, \mathbb{Q}_p)$, $* = 0, 1$, $R/pR = A$;
- Tsygan, Petrov-Vologodsky (2019): A a (dg-)algebra over $\mathbb{F}_p$ with a choice of $\mathbb{Z}_p$-algebra lifting R, then $\widehat{\mathrm{HP}(R)} \cong \widehat{\mathrm{TP}(A)}$. The right hand side is crystaline cohomology under suitable assumptions;
- Lack of homotopy invariance in the latter approach makes it impossible to compute.

Analytic cyclic homology

Theorem (Cortiñas-Meyer-M, 2020)

We construct a functor

$$\mathrm{HA}\colon \{\textit{Complete, bornologically torsion-free, } \mathbb{Z}_p\textit{-algebras}\} \longrightarrow \{\mathbb{Z}_2 \textit{ - graded } \mathbb{Q}_p\textit{-vector spaces}\}$$

that satisfies that satisfies homotopy invariance, excision, Morita invariance.

Theorem (Meyer-M, 2022)

We construct a functor

$$\mathrm{HA}\colon \{\mathbb{F}_p\textit{-algebras}\} \longrightarrow \{\mathbb{Z}_2 \textit{ - graded } \mathbb{Q}_p\textit{-vector spaces}\}$$

that satisfies the above properties, and is independent of choice of lifting to $\mathbb{Z}_p$-algebra.

Local cyclic homology

- The independence result requires liftings to satisfy certain finiteness assumptions mod p;
- There is a way to functorially associate to any lifting a nuclear bornological algebra that satisfies these finiteness assumptions automatically;
- If the original algebra D is dagger, then the nuclear algebra $N(D)$ associated to it is still dagger.

Theorem

For a Banach algebra B, define $\mathrm{HL}(B) = \mathrm{HA}(N(B))$; this functor satisfies all the properties of analytic cyclic homology. Furthermore for R a V-subalgebra of a Banach algebra B such that $R/pR = B/pB$, we have

$$\mathrm{HA}(R^\dagger) = \mathrm{HL}(R^\dagger) = \mathrm{HL}(B).$$

Algebras of interest in the nonarchimedean setting - I

- The analogue of continuous functions on a topological space is analytic functions on a rigid analytic space;
- Formally, consider $T_n := \{\sum_I c_I x^I \in \mathbb{Z}_p[[x_1, \dots, x_n]] : |c_I| \to 0\}$. This is a Banach $\mathbb{Z}_p$-algebra with the norm $\|\sum c_I x^I\| = \max |c_I|$;
- This is a Noetherian Banach algebra, so every ideal I is closed. The quotient T_n/I plays the role of $C_0(X)$ in rigid analytic geometry.

Algebras of interest in the nonarchimedean setting - II

- The analogue of a graph C^*-algebra is the p-adic completion of a Leavitt path algebra of a graph;
- Crossed product algebras of finite group actions on varieties over $\mathbb{F}_p$;
- The analogue of a group C^*-algebra of a locally compact group or discrete group is the p-adic completion of the group ring;
- Bounded operators on a p-adic Hilbert space.

Algebras of interest in the nonarchimedean setting - III

Let G be a p-adic Lie group or an algebraic group.

- Locally analytic functions $C^{\mathrm{an}}(G, V)$ on a p-adic Lie group with values in a Banach $\mathbb{Q}_p$-vector space;
- Distribution algebras $D^{\mathrm{an}}(G, V) = C^{\mathrm{an}}(G, V)'$ with convolution product;
- Completions of quantised coordinate algebras $\widehat{\mathcal{O}_q(G)}$ and quantised enveloping algebras $\widehat{U_q(\mathfrak{g})}$ for $0 \neq q \in \mathbb{Q}_p$. (Soibelman, Schneider-Titelbaum, Dupré)

(Multiplier) Hopf algebras

- Function algebras and their duals (convolution algebras) are multiplier Hopf algebras in the sense of van Daele;
- These are algebras with a coassociative comultiplication into the multiplier algebra $\Delta \colon A \to \mathcal{M}(A)$ such that the canonical Galois maps are isomorphisms into $A \otimes A$;
- The category of regular multiplier Hopf algebras with nonzero invariant functionals includes compact and discrete quantum groups. They generalise the Pontrjagin duality for compact and discrete abelian groups to Hopf algebras;
- Purely algebraic theory; does not include topological information.

Bornological quantum groups

- Replace the category of modules (over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, or any Banach ring) to complete bornological modules over a Banach ring;
- This is a closed, symmetric monoidal, complete and cocomplete, quasi-abelian category;
- Generalises the theory of multiplier Hopf algebras to include analytic objects (intrinsically).

Definition

A bornological quantum group is an (essential) bornological algebra with a comultiplication $\Delta \colon A \to \mathcal{M}(A \overline{\otimes} A)$ and a left invariant functional such that the Galois maps are isomorphisms.

Examples of bornological quantum groups

- Any algebraic quantum group over $\mathbb{Z}_p$ or $\mathbb{Q}_p$;
- G a compact p-adic Lie group, $\mathrm{C}(G, \mathbb{Q}_p) \supseteq \mathrm{C}^{\mathrm{an}}(G, \mathbb{Q}_p) \supseteq \mathrm{C}^{\infty}(G, \mathbb{Q}_p)$;
- For G as above, $D(G, \mathbb{Q}_p)$, $D^{\mathrm{an}}(G, \mathbb{Q}_p)$, $D^{\infty}(G, \mathbb{Q}_p)$;
- G as above or a finitely generated, discrete group, $\mathbb{Z}_p[G]$, $\mathbb{Z}_p[G]^{\dagger}$ and $\widehat{\mathbb{Z}_p[G]}$;
- The algebra $U_q(\mathfrak{g})$ is a bornological Hopf algebra - its a special case of a certain double-bosonisation construction.

Hochschild homology of smooth convolution algebras - I

- Invariants of smooth subalgebras are more tractable than their Banach algebraic counterparts;
- For totally disconnected and discrete groups with word-length functions satisfying certain growth conditions, one can use coarse geometry techniques to reduce homological properties of smooth group algebras to those of the incomplete group ring;
- More precisely, let $A = \mathbb{C}[G]$ and $P_\bullet \to A$ be a certain projective resolution by A-bimodules. It is derived from the complex computing the group cohomology of G;
- Then for $B = \mathcal{S}(G)$, $B \overline{\otimes}_A P_\bullet \overline{\otimes}_A B \to B$ is a projective B-bimodule resolution of B (Meyer).

Hochschild homology of smooth convolution algebras - II

- Algebras pairs $A \to B$ such that projective resolutions of A extend by base-change to projective resolutions of B are called isocohomological embeddings.
- By the same techniques as in the complex case, $\mathbb{Q}_p[G] \to \mathbb{Q}_p[G]^\dagger$ is an isocohomological embedding;
- The assumptions on G help us find a contracting homotopy on $C_n(G)$, which extends to a bounded contracting homotopy on the complex $\mathbb{Q}_p[G]^\dagger \overline{\otimes}_{\mathbb{Q}_p[G]} C_n(G)$;
- This is used to show that $\mathbb{Q}_p[G]^\dagger \overline{\otimes}^{\mathbb{L}}_{\mathbb{Q}_p[G]} \mathbb{Q}_p[G]^\dagger \cong \mathbb{Q}_p[G]^\dagger$.

Summary

- Smooth subalgebras of group C^*-algebras or Banach algebras are a source of bornological quantum groups;
- We extend this notion to "smooth subalgebras" (ie, dagger subalgebras) of non-archimedean Banach algebras;
- Local cyclic homology does not see the difference between these categories by depending only on the reduction mod p;
- It is easier to compute invariants in the category of smooth or dagger algebras.